

Metric ultraproducts of finite groups with respect to some length functions

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Abstract We consider metric ultraproducts of finite groups with respect to some classes of length functions. All sofic groups embed into these ultraproducts. We study embeddings of normed groups. We also show that in some natural situations such an ultraproduct is a simple group.

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1 Introduction

Let G be a group. A function $l : G \rightarrow [0, \infty)$ is called a *pseudo length function* if

- (i) $l(1) = 0$;
- (ii) $l(g) = l(g^{-1})$;
- (iii) $l(gh) \leq l(g) + l(h)$.

A *length function* is a pseudo length function satisfying

$$(i') \quad l(g) = 0 \text{ if and only if } g = 1, \text{ where } g \in G.$$

A pseudo length function is *invariant* if $l(h^{-1}gh) = l(g)$ for all $g, h \in G$. In this case it defines an invariant pseudometric by $l(gh^{-1})$. It becomes metric

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if l is a length function. In this case we say that G is a *normed group*. We consider normed groups as metric groups too.

Metric ultraproducts of normed groups of bounded diameter, say r , are defined as follows. Let (G_i, l_i) , $i \in I$, be a family of groups equipped with invariant length functions and let Φ be an ultrafilter on I . Then

$$N = \{(x_i)_{i \in I} \in \prod_I G_i : \lim_{i \rightarrow \Phi} l_i(x_i) = 0\}$$

is a normal subgroup of $\prod_I G_i$. The *metric ultraproduct* $\prod_I (G_i, l_i) / \Phi$ is defined to be $(\prod_I G_i) / N$ where the length function is defined by

$$l(xN) = \lim_{i \rightarrow \Phi} l_i(x_i).$$

The latter is well-defined by compactness of $[0, r]$. This definition corresponds to Section 2.4 from [9].

Metric ultraproducts of finite normed groups are deserved a particular attention in group theory. This is mainly motivated by investigations of *sofic groups*. We remind the reader that a group G is called *sofic* if G embeds into a metric ultraproduct of finite symmetric groups with the Hamming distance [9]. A group G is called *hyperlinear* if G embeds into a metric ultraproduct of finite-dimensional unitary groups with the normalized Hilbert-Schmidt metric [9]. It is an open question whether these classes are the same and whether any countable group is sofic/hyperlinear.

If in the definition of metric ultraproducts we do not assume diameter boundedness, we arrive to a more general construction, see [9], Sections 2.3 and 2.4, and to the following notion. A group G is called *weakly sofic* if G embeds into a metric ultraproduct of finite groups with invariant length functions [6]. It is not known if this class coincides with the former ones.

Thus the following general question becomes very interesting: *how strongly do properties of metric ultraproducts depend on particular choice of length functions?* In particular, *describe metric ultraproducts of familiar classes of finite groups with various invariant length functions.*

It is worth noting that papers [1], [11] and [12] already suggest considering classes of finite groups with some special length functions. This in particular produces new versions of soficity.

In our paper we concentrate on some natural modifications of the length function from the paper of A.Stolz and A.Thom [11]. On the one hand they are sufficiently general to cover all sofic groups. On the other hand some statements of [11] still hold with respect to them. Moreover we discover that in some respects these length functions resemble the Hamming length in symmetric groups. In Section 3 we prove that in the case of classical groups of unbounded dimension the corresponding metric ultraproducts are always simple groups. This generalises the corresponding theorem from [4] concerning symmetric groups with Hamming distance.

When a metric ultraproduct $\prod(G_i, l_i)/\Phi$ is a simple group there is no δ such that all elements of length $< \delta$ form a proper subgroup. Thus it becomes an interesting question if one can correct length functions l_i so that the ultraproduct gets such a subgroup. Moreover given G , a finitely generated subgroup of $\prod(G_i, l_i)/\Phi$, *what norms can G inherit under such embeddings?* We will discuss this in Sections 2 and 4. In Section 4 we in particular show that under some mild assumptions on a subgroup $G_1 < G$ we can shorten the norms of elements of G_1 in such embeddings. Moreover we can do it using our modifications of length functions from [11].

We continue this introduction in Section 2.2 after some definitions and easy observations of Section 2.1.

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2 Pseudo length functions

2.1 Length corrections

The following lemma is obvious.

Lemma 2.1 *Let $H < G$. For any invariant (pseudo) length function l on G the restriction of l to H is an invariant (pseudo) length function.*

The following statements are taken from the paper of A.Stolz and A.Thom [11] (Lemmas 2.1 and 2.2).

Lemma 2.2 (i) *If G is a finite group with an invariant (pseudo) length function l and H is a normal subgroup of G , then*

$$l_{G/H}(gH) = \inf\{l(gh) : h \in H\}$$

defines an invariant (pseudo) length function on G/H .

(ii) *If G is a finite group and H is a normal subgroup of G so that G/H has an invariant pseudo length function l , then*

$$l^G(g) = l(gH)$$

is an invariant pseudo length function on G .

Using this lemma we now describe some ways of correction of length functions.

Lemma 2.3 *Let G be a finite group, H be normal in G and l be an invariant (pseudo) length function bounded by r . Then*

(i) for any natural $k \geq r$ and s the following function l_H^{*sk} is an invariant (pseudo) length function on G bounded by 1:

$$\text{if } g \in H \text{ then } l_H^{*sk}(g) := \frac{\sqrt[s]{l(g)}}{k+1} \text{ and if } g \notin H \text{ then } l_H^{*sk}(g) := 1.$$

(ii) if $r = 1$, then for any natural numbers s, s' and t the function

$$l_H^{ss't}(g) = \frac{t}{t+1} \sqrt[t]{l_{G/H}(gH)} + \frac{\sqrt[s']{l(g)}}{t+1}$$

is an invariant (pseudo) length function bounded by 1.

Proof. The proof is straightforward; one should use the inequality $\sqrt[s]{a+b} \leq \sqrt[s]{a} + \sqrt[s]{b}$. The boundedness follows from the inequality $l_H^{ss't} \leq \max(\sqrt[s]{l}, \sqrt[s']{l})$. \square

Let \mathcal{G}_1 and \mathcal{G}_2 be two countable families of groups with pseudo length functions. We assume that \mathcal{G}_1 and \mathcal{G}_2 are defined on the same class of groups, i.e. for any group G

$$\text{there is } l_1 \text{ with } (G, l_1) \in \mathcal{G}_1 \Leftrightarrow \text{there is } l_2 \text{ with } (G, l_2) \in \mathcal{G}_2.$$

Let us enumerate all groups of this form: G_0, G_1, G_2, \dots . We say that \mathcal{G}_1 is *asymptotically bounded* by \mathcal{G}_2 (with respect to our enumeration) if there are constants c and n_0 so that for every $n > n_0$ and every choice of elements $g \in G_n$ we have $l_1(g) \leq c l_2(g)$ (see [11]). We call these classes *asymptotically equivalent* if they are asymptotically bounded with respect to each other.

We say that \mathcal{G}_1 is *asymptotically bounded* by \mathcal{G}_2 *up to a polynomial* if there is a constant c and natural numbers $m > 0$ and n_0 so that for every $n > n_0$ and every choice of elements $g \in G_n$ we have

$$(l_1(g))^m \leq c \cdot l_2(g) \text{ if } l_1(g) < 1 \text{ and}$$

$$(l_1(g)) \leq c \cdot l_2(g)^m \text{ if } l_1(g) \geq 1.$$

The following lemma is obvious.

Lemma 2.4 *Let $l \leq 1$ be a generic pseudo length function on an enumerated family of all pairs of finite groups (G, H) where H is normal in G (and l is defined on G).*

Then for any fixed t the pseudo length functions l and l_H^{11t} are asymptotically equivalent:

$$l \leq (t+1)l_H^{11t} \text{ and } l_H^{11t} \leq l.$$

On the other hand for any natural numbers s, s'

$$l \leq (t+1)l_H^{ss't} \text{ and } (l_H^{ss't})^m \leq l \text{ where } m = \max(s, s'),$$

i.e. the pseudo length functions l and $l_H^{ss't}$ are asymptotically equivalent up to a polynomial.

The main object of our paper is the following pseudo length function.

Definition 2.5 *If G is a finite group then the function*

$$\text{conjugacy length } l_c(g) = \frac{\log|g^G|}{\log|G|}$$

is an invariant pseudo length function; it defines an invariant pseudometric by $l_c(g_1g_2^{-1})$ (Proposition 2.3 of [11]).

By Proposition 2.4 of [11] the function l_c is a length function exactly when G has trivial centre.

Let H be a normal subgroup of G . Applying the lemmas above we obtain that

the functions $(l_c)_H^{*sk}$ and $(l_c)_H^{ss't}$ are invariant pseudo length function.

We will see in Section 4 that they are very helpful.

The same construction can be applied to some other pseudo length functions. The following invariant length function is taken from [6]. It will be used in the proof of Theorem 4.2 below. Consider a countable class of groups \mathcal{C} where every group G is considered as a pair together with a finite distinguished set Δ of conjugacy classes. Then the conjugacy graph $\Gamma(G, \Delta)$ is defined on the set V of all conjugacy classes of G as follows: a pair $(x, y) \in V \times V$ is an edge if for some $c \in \Delta$ we have $x \subset cy$. We assume that for any $(G, \Delta) \in \mathcal{C}$ the diameter of the connected component containing e_G is finite.

Define $l_\Delta(g)$ to be the distance from e_G to g^G in $\Gamma(G, \Delta)$.

When e_G and g^G belong to distinct connected components of the conjugacy graph we assume that $l_\Delta(g)$ is the next natural number after the diameter of the component of e_G . This function can be normalised. If G is finite and H is the normal subgroup of G generated by Δ , then the definition of l_Δ slightly resembles $(l_c)_H^{*sk}$.

Remark 2.6 Assume that $G = H$ and S is a finite symmetric set representing all classes of Δ so that G is generated by S . The *cancelation length* of a word w in the alphabet S is defined to be the least number of letters to be deleted from w in order to obtain a word trivial in G . The *cancelation norm* of an element $g \in G$ is defined to be the minimal cancelation length of a representing word. By Proposition 2.A of [2] it coincides with l_Δ .

2.2 Subgroups of metric ultraproducts

Consider a metric ultraproduct $\prod_I(G_i, l_i)/\Phi$ of finite normed groups. It is easy to see that replacing each l_i by $\frac{l_i}{1+l_i}$ the ultraproduct becomes the same group with respect to a length function < 1 . Below we will usually assume that length functions are bounded by 1.

Assuming that for every i , H_i is a normal subgroup of G_i we may replace l_i by $(l_i)_{H_i}^{*11}$. As a result we obtain a new ultraproduct where all elements of length $< \frac{1}{2}$ form a normal subgroup. In particular it can be non-isomorphic with $\prod_I(G_i, l_i)/\Phi$ as an abstract group. For example this is the case for ultraproducts of classical groups with respect to the conjugacy length. This obviously follows from the main theorem of Section 3 that when dimension is not bounded these ultraproducts are simple groups. Thus taking G_i as $PGL_{n_i}(q_i)$ and taking H_i as the corresponding $PSL_{n_i}(q_i)$ we obtain a nice illustration of the observation above.

Let P be a finitely generated group which embeds into $\prod_I(G_i, l_i)/\Phi$. It can happen that the length correction above forbids an embedding of P into the metric ultraproduct after the correction.

How can we correct the length functions so that P still a subgroup of the corresponding ultraproduct, but its length function becomes completely different? In particular let P_1 be a distinguished normal subgroup of P . Can we arrange that P_1 consists of elements of length $< \frac{1}{2}$? This question will be studied in Section 4.

3 Simplicity of metric ultraproducts

Let P be a metric ultraproduct of normed groups of a class \mathcal{C} . If \mathcal{C} is the class of all finite simple groups with the conjugacy length then by Theorem 3.1 of [11] the group P is simple. Consider the situation when \mathcal{C} consists of groups G with trivial centre containing a subgroup H which is simple. The best example is the class \mathcal{S} of all finite symmetric groups S_n with the subgroups A_n of even permutations. It is shown in [4] that any infinite metric ultraproduct of groups from \mathcal{S} with respect to Hamming's metrics is still a simple group. In this section we obtain the same result in some other cases. We start with the following logic lemma.

Lemma 3.1 *Let \mathcal{C} consist of finite normed groups G with trivial centre and bounded norms. We assume that for every $\varepsilon > 0$ there is a natural number m such that any $G \in \mathcal{C}$ contains a simple subgroup H which coincides with $(h^H)^m$ for any $h \in H$ with $l(h) > \varepsilon$.*

Assume that for any sequence $g_i \in G_i \in \mathcal{C}$ there is a sequence $h_i \in G_i$ such that $\lim_{i \rightarrow \infty} l(h_i) = 0$, and all $g_i h_i$ belong to H_i .

Then any infinite metric ultraproduct of G_i is a simple group.

Proof. Let Φ be an ultrafilter on ω such that the ultraproduct $\prod G_i/\Phi$ is infinite. For any sequence $g_i \in G_i \in \mathcal{C}$ such that under Φ the limit $\lim_{i \rightarrow \infty} l_c(g_i)$ exists and is greater than 0, there is a sequence $g'_i \in G_i$ such that $\lim_{i \rightarrow \infty} l(g'_i) = 0$, and all $g_i g'_i$ belong to H_i . Thus any non-trivial element of the ultraproduct of G_i can be presented by a sequence h_i which belongs to the ultraproduct of the corresponding $H_i < G_i$.

Assume that $g \in \prod G_i/\Phi$ is represented by $(g_i)_{i \in \omega}$ and $(h_i)_{i \in \omega}$ as above and $\varepsilon < \lim_{i \rightarrow \Phi} l(g_i)$. Choose m as in the formulation of the lemma. Since $(h_i^{H_i})^m = H_i$ with respect to Φ , we see that

$$(g^{\prod G_i/\Phi})^m = \prod G_i/\Phi.$$

□

We now show how this lemma works in some very natural cases. We will apply it together with a very strong result from [8]. Theorem 1.1 of [8] states the existence of a universal constant c such that whenever H is a finite non-abelian simple group, $g \in H \setminus \{1\}$ and $m \geq c \frac{\log |H|}{\log |g^H|}$ we have $(g^H)^m = H$.

Let V be an n -dimensional vector space over a finite field K with $|K| = q$. Let f be a sesquilinear form on V , write G for the group of K -linear maps preserving f , write Z for the centre of G and write H for the derived group of G . We are concerned with the cases when f is one of the following: the zero form, a nondegenerate symplectic form, a nondegenerate hermitian form with respect to a non-identity involutory automorphism J of K , or a nondegenerate symmetric form (for odd q). There is an additional case when q is even, we then consider G as a group of linear maps preserving a non-singular quadratic form Q with $f(x, y) = Q(x + y) - Q(x) - Q(y)$. These are *classical groups* over finite fields.

In all these cases the groups HZ/Z are simple except for small values of $\dim V$ and $|K|$.

Describe the metric ultraproducts of groups G/Z with respect to natural invariant length functions which are simple.

It is worth noting that by Proposition 2.7 of [13] the ultraproducts of groups G/Z with respect to the $\{0, 1\}$ -metric are not simple unless $G = H$. When the groups are normed by the conjugacy length the situation is different. The first statement of the following theorem is in fact proved in [4].

Theorem 3.2 *Let \mathcal{S} be the family of all symmetric groups S_n and let \mathcal{G} be the class of all projective classical groups G/Z over finite fields of dimensions > 4 . We consider groups of both classes as normed groups with respect to the conjugacy length. Then*

- (i) *any infinite metric ultraproduct of members of \mathcal{S} is a simple group;*
- (ii) *any non-discrete metric ultraproduct of members of \mathcal{G} is a simple group;*

any discrete metric ultraproduct of members of \mathcal{G} is a Chevalley group over a standard ultraproduct of finite fields.

The same statements hold for these classes with respect to length functions $(l_c)_{A_n}^{ss't}$ and the corresponding subgroups A_n and HZ/Z respectively (where s , s' and t are fixed).

Proof. (i) By Theorem 2.16 of [11] the conjugacy length is asymptotically equivalent to the Hamming's distance in \mathcal{S} as well as in \mathcal{A} , the family of finite alternating groups A_n . Now note that the Hamming's norm of any transposition (i, j) is $\frac{2}{n}$. Thus the conjugacy length of a transposition from S_n converges to 0 when $n \rightarrow \infty$. This in particular means that for any sequence $g_i \in G_i \in \mathcal{S}$ such that the metric ultraproduct of G_i is infinite and under the corresponding ultrafilter $\lim_{i \rightarrow \infty} l_c(g_i)$ exists and is greater than 0, there is a sequence $g'_i \in G_i$ such that $\lim_{i \rightarrow \infty} l(g'_i) = 0$, and all $g_i g'_i$ are even permutations. Thus any element of the ultraproduct of G_i can be presented by a sequence h_i which belongs to the ultraproduct of the corresponding $A_n < S_n$.

By Theorem 1.1 of [8] there is a universal constant c such that for any $H \in \mathcal{A}$, $g \in H$ and $m \geq c \frac{\log|H|}{\log|g^H|}$ we have $(g^H)^m = H$. Since $|S_n| = 2|A_n|$ and $|g^{S_n}| \leq 2|g^{A_n}|$, the equality $(g^H)^m = H$ holds for $H = A_n$ under the assumption that

$$m \geq 2c \frac{\log|S_n|}{\log|g^{S_n}|} \text{ (which is } \frac{2c}{l_c(g)}).$$

Thus to apply Lemma 3.1, given $\varepsilon > 0$ take $m \geq \frac{2c}{\varepsilon}$. Then the sequence (g_i) as above defines an element of the metric ultraproduct such that the m -th power of its conjugacy class covers the group.

By Lemma 2.4 this argument works for the length functions $(l_c)_{A_n}^{ss't}$.

(ii) Consider a normed ultraproduct $\prod_{j \in J} G_j / \mathcal{U}$. Since \mathcal{U} is an ultrafilter and \mathcal{G} divides into finitely many classical series, we may assume that all G_j belong to one of them.

Let us start with the case of ultraproducts from \mathcal{PGL} , the class of all projective linear groups $PGL_n(q)$ over finite fields \mathbf{F}_q with $n > 4$. We show that any non-discrete metric ultraproduct of members of \mathcal{PGL} is a simple group.

By Theorem 2.20 of [11] in the class of all $PSL_n(q)$ and in the class of all $PGL_n(q)$ the conjugacy length is asymptotically equivalent to the *Jordan length* l_J , where

$$l_J(g) = n^{-1} \inf_{a \in F_q^*} rk(a - g).$$

Note that the Jordan length does not depend on q , in particular the minimal norm of a non-trivial element from $PGL_n(q)$ is always $\frac{1}{n}$. This means that if the metric ultraproduct of groups $G_i \in \mathcal{PGL}$ is not a discrete space then the dimension of these groups is not bounded with respect to \mathcal{U} .

Now note that the Jordan length of a diagonal $n \times n$ -matrix $\text{diag}(1, 1, \dots, 1, d)$ is $\frac{1}{n}$. Thus the conjugacy length of such a matrix from $PGL_n(q)$ converges to 0 when $n \rightarrow \infty$. This in particular means that for any sequence $g_i \in G_i \in \mathcal{PGL}$ such that the metric ultraproduct of G_i is non-discrete and $\lim_{i \rightarrow \mathcal{U}} l_c(g_i)$ is greater than 0, there is a sequence of diagonal $h_i \in G_i$ such that $\lim_{i \rightarrow \mathcal{U}} l(h_i) = 0$, and all $g_i h_i$ are from the corresponding $PSL_n(q)$ -s. Since $\frac{\log(q)}{\log|PGL_n(q)|} \rightarrow 0$ with respect to \mathcal{U} and $\lim_{i \rightarrow \mathcal{U}} l_c(g_i h_i)$ is greater than 0, we have that

$$\frac{\log(q)}{\log|(g_i h_i)^{PSL_n(q)}|} \rightarrow 0 \text{ under } \mathcal{U}.$$

By Theorem 1.1 of [8] there is a universal constant c such that for any $H \in \mathcal{PSL}$, $g \in H$ and $m \geq c \frac{\log|H|}{\log|g^H|}$ we have $(g^H)^m = H$. Using the observations above, as in part (i) we obtain that the sequence $(g_i h_i)$ defines an element of the metric ultraproduct such that for some m the m -th power of its conjugacy class covers the group. In fact to apply the argument of the proof of Lemma 3.1 it suffices to take $m \geq \frac{2c}{l_c(g_i h_i)}$ with respect to \mathcal{U} (computing l_c in G_i), i.e.

$$m \geq 2c \frac{\log(q) + \log|H|}{\log(q) + \log|(g_i h_i)^H|} \text{ with respect to } \mathcal{U}.$$

If the metric ultraproduct of groups G_i with respect to the Jordan length is infinite and discrete, then there is n such that all G_i are of the form PGL_n up to the ultrafilter. As in this case the metric ultraproduct coincides with the non-metric one, we may apply the results of [10]. By Proposition 1 of that paper and some folklore observations, the ultraproduct of $PGL_n(K_i)$ (resp. $PSL_n(K_i)$) is PGL_n (PSL_n) over the corresponding ultraproduct of fields.

The use of the Jordan length above can be replaced by some additional computations. Such arguments will be applied below.

Let us consider the cases of non-trivial bilinear forms (by Proposition 3.1 from [11] we may omit the case when $G/Z = HZ/Z$, i.e. for example where f is symplectic over an odd field). By Section II.3 of [3] the determinant of any unitary transformation over a field K is of the form $\gamma^J \gamma^{-1}$ where $\gamma \in K$. This in particular means that multiplying any matrix from $U_n(K)$ by an appropriate diagonal matrix of the form $\text{diag}(1, 1, \dots, 1, \gamma^J \gamma^{-1})$ we obtain an element from $SU_n(K)$.

The conjugacy length of such a diagonal matrix can be computed as follows. Writing the matrix as $e + (d - 1)e_{nn}$ with $d = \gamma^J \gamma^{-1}$, we see that any its conjugate is of the form $e + (d - 1)(\vec{c})' \vec{b}$ where \vec{b} and \vec{c} are n -vectors so that $\vec{b}(\vec{c})' = 1$. Since the unitary space over \mathbf{F}_{q^2} always has an orthonormal basis and any matrix from $U_n(q^2)$ takes an orthonormal bases to an orthonormal one, we have $\vec{b} = \vec{c}^J$. In particular the size of the conjugacy class $\text{diag}(1, 1, \dots, 1, \gamma^J \gamma^{-1})$ in $U_n(q^2)$ is between q and q^{2n} . Since by Section II.6 of [3]

$$|U_n(q^2)| = (q^n - (-1)^n)q^{n-1}(q^n - (-1)^{n-1})q^{n-2} \cdot \dots \cdot (q^2 - 1)q(q + 1),$$

the conjugacy length of $\text{diag}(1, \dots, 1, \gamma^J \gamma^{-1})$ is asymptotically equivalent to $\frac{1}{n}$. This means that if the metric ultraproduct of groups G_i of the form U_n is a discrete space then the dimension of these groups is bounded with respect to \mathcal{U} .

A non-central element $g \in U_n$ with the minimal conjugacy length has the maximal size of its centraliser. We decompose g into a product of commuting semisimple and unipotent elements and consider their Jordan forms. As in Section 7 of [8] we obtain a decomposition of the matrix of g into a sum of diagonal matrices, Jordan blocks and tensor products of Jordan blocks with irreducible matrices. When the decomposition is non-trivial, the index of $C_{U_n}(g)$ in U_n is clearly greater than q . Since by Lemma 4.4 of [8] the size of the centraliser of the Jordan k -block is not greater than q^{2k} , we also have $|U_n : C_{U_n}(g)| \geq q$ when g is a Jordan block or a tensor product. Thus the conjugacy length of g does not depend on q , i.e. the minimal norm of a non-central element from $U_n(q^2)$ converges to 0 only when $n \rightarrow \infty$.

This already means that if the metric ultraproduct of groups G_i of the form PU_n is not a discrete space then the dimension of these groups is not bounded with respect to \mathcal{U} . Thus we may apply the same argument as above (using an appropriate part of Theorem 1.1 of [8]).

In the orthogonal case the determinant is 1 or -1 . In general the derived subgroup Ω_n is a proper subgroup of O_n^+ (of determinant 1). The structure of 2-subspaces of V becomes crucial in this case. A 2-subspace P is called a *hyperbolic plane* if it has a basis v_1 and v_2 consisting of singular vectors (i.e. $Q(v_i) = 0$) with $f(v_1, v_2) = 1$. In the case of characteristic $\neq 2$ (when Q is determined uniquely by f) we will use the following fact from Section II.8 of [3].

Let P be a hyperbolic plane in V . Then any element g of O_n can be presented as a product sw with $w \in \Omega_n$ so that s fixes the orthogonal complement of P .

Such an s is determined by an orthogonal transformation of P . Thus the size of the conjugacy class of s in $O_n(q)$ is bounded by $q^n q^{n-1}$. On the other hand by Section II.9 of [3] for odd n ,

$$|O_n^+(q)| = (q^{n-1} - 1)q^{n-2}(q^{n-3} - 1)q^{n-4} \cdot \dots \cdot (q^2 - 1)q,$$

and for $n = 2m$,

$$|O_n^+(q)| = (q^{2m-1} - \varepsilon q^{m-1})(q^{2m-2} - 1)q^{2m-3} \cdot \dots \cdot (q^2 - 1)q, \text{ with } \varepsilon \in \{-1, 1\}.$$

Thus the conjugacy length of s in $O_n(q)$ converges to 0 when $n \rightarrow \infty$. In particular, the arguments above also work in this case (using appropriate places of [8] and [10]).

In the case of characteristic 2 an orthogonal space is determined by a quadratic form. In this case $f(x, y)$ is symplectic of even dimension. Moreover

there are two kinds of quadratic geometry on V : an O^{+1} -geometry, where V is an orthogonal sum of hyperbolic planes, and an O^{-1} -geometry, where V is an orthogonal sum of $\frac{n}{2} - 1$ hyperbolic planes and one *definite* (non-singular) plane. In both cases the statement on decomposition $g = sw$, with $w \in \Omega^\varepsilon$ also holds ([3], Section II.10). Since in the first case

$$|O_{2m}^+(q)| = (q^m - 1)(q^{2(m-1)} - 1)q^{2(m-1)} \cdot \dots \cdot (q^2 - 1)q^2,$$

and in the second one

$$|O_{2m}^+(q)| = (q^m + 1)(q^{2(m-1)} - 1)q^{2(m-1)} \cdot \dots \cdot (q^2 - 1)q^2,$$

we can apply our arguments above.

By Remark 2.4 the arguments above also work for the length functions of the fom $(l_c)_{PSL_n}^{ss't}$ and their relatives. \square

Remark 3.3 It is worth noting that if we consider the classes \mathcal{S} and \mathcal{PGL} with respect to the family of all length functions $(l_c)_H^{ss't}$ for all natural s, s' and t (with respect to \mathcal{A} and \mathcal{PSL}), then Theorem 3.2 does not remain true. For example if we consider S_n with respect to $(l_c)_{A_n}^{n,1,3}$, then the norm of a transposition (i, j) can be evaluated as $\frac{3\sqrt[3]{2d}}{4\sqrt[3]{n}} + \frac{d}{2n}$ for some constant d (replace the conjugacy length by Hamming length and apply Lemma 2.4). In particular for sufficiently large n the norm of any element of $S_n \setminus A_n$ is greater than $\frac{1}{2}$. On the other hand by the definition of $(l_c)_{A_n}^{n,1,3}$ the norm of any element of A_n is less than $\frac{1}{4}$. This in particular shows that the elements of the metric ultraproduct of $(S_n, (l_c)_{A_n}^{n,1,3})$ which have norm $< \frac{1}{4}$, form a non-trivial normal subgroup.

4 Embeddings of weakly sofic groups and LEF-groups

Let P be a finitely generated subgroup of a metric ultraproduct $\prod_I(G_i, l_i)/\Phi$ of finite normed groups (i.e. P is weakly sofic). Let P_1 be a distinguished normal subgroup of P so that P/P_1 is residually finite. Using the approach of Section 2 we now show that P can be embedded into a metric ultraproduct \hat{P} of finite normed groups so that P_1 consists of all elements of P which have norms $\leq \frac{1}{2}$. Moreover we will show that when P is a *LEF*-group [5], the norms of that finite groups can be taken in the form $(l_c)^{*st}$, introduced in Section 2.

For convenience of the reader we recall the following definition.

Definition 4.1 Let \mathcal{C} be a class of normed groups. A group G is said to have the **\mathcal{C} -approximation property** [12] if for any $g \in G$ there exists $\delta_g > 0$

such that for all finite $D \subset G$ and $\varepsilon > 0$ there exists a group $(C, l) \in \mathcal{C}$ and a map $\phi : G \rightarrow C$ so that

$$\phi(e) = e, \quad l(\phi(g)) \geq \delta_g \text{ for all } g \in D \text{ and}$$

$$l(\phi(gh)\phi(h)^{-1}\phi(g)^{-1}) < \varepsilon \text{ for all } g, h \in D \text{ with } gh \in D.$$

Proposition 1.8 of [12] states that a countable group has the \mathcal{C} -approximation property if and only if G embeds into a metric ultraproduct of members of \mathcal{C} with respect to a non-principal ultrafilter on ω .

Theorem 4.2 *Let $N \triangleleft N_1 \triangleleft F$, where F is a finitely generated free group. Let $P = F/N$ and $P_1 = N_1/N$. Assume that F/N is weakly sofic and F/N_1 is residually finite. Then P embeds into a metric ultraproduct of finite groups with norms ≤ 1 so that P_1 consists of all elements of P which have norms $\leq \frac{1}{2}$ in that ultraproduct.*

Proof. The proof is based on results and methods of [6]. It also uses the idea of length corrections of Section 2. Let $\varepsilon < \frac{1}{4}$. We want to apply an appropriate version of the approximation property for P and $\frac{10}{9}\varepsilon$ where $\delta_g = \frac{9}{40}$. We additionally arrange that all elements of P_1 have norms $\leq \frac{1}{2}$ and elements of $P \setminus P_1$ are of norms $\geq \frac{21}{40}$.

The resulting (C, l) from the formulation will be denoted (G, \tilde{l}) below. Let D be a finite subset of F/N presented by words $\{w_1, \dots, w_k\} \subset F$. We may assume that the empty word belongs to D . Let $r = 3\max(|w_i| : i \leq k)$.

Since P is weakly sofic, by Theorem 4.3 of [6] for any u_1, \dots, u_m from N the profinite closure of $u_1^F \cdot \dots \cdot u_m^F$ is contained in N . Thus there is a homomorphism ϕ from F to a finite group G so that for any $w \in D^3 \setminus N$ the element $\phi(w)$ does not belong to any product $\phi(u_1)^G \cdot \dots \cdot \phi(u_m)^G$ with $u_i \in N \cap D$ and $m \leq \frac{9}{10\varepsilon}$. Let H be a normal subgroup of G defined by $H = \phi(N_1)$. Since F/N_1 is residually finite we may suppose that $\phi(D \setminus N_1) \cap H = \emptyset$.

Repeating the sufficiency argument of Lemma 6.4 of [6] we build an invariant length function l on G so that all elements of $\phi(D^3 \cap N)$ have norm $\leq \varepsilon$ but all elements of $\phi(D \setminus N)$ have norm $\geq \frac{9}{10}$. For convenience of the reader we recall that this is εl_Δ (see Section 2) where Δ consists of conjugacy classes $\phi(w)^G$ with $w \in N$ and $|w| \leq r$.

Let us define an invariant length function \tilde{l} on G . We start with a preliminary function $\gamma(x)$ on \mathbb{R}^+ . Let $m_0 = \max(l(g) : g \in G)$. Let $s > 1$ satisfy $\sqrt[s]{m_0} < 2$. Let $\gamma(x)$ be a continuous function which equals $\sqrt[s]{x}$ for real numbers $> \frac{9}{10}$ and is of the form $p \cdot x$ for $x \leq \frac{9}{10}$. It is clear that $1 \leq p \leq \frac{10}{9}$. It is worth noting that $\gamma(x+y) \leq \gamma(x) + \gamma(y)$.

For elements $g \in H$ we define

$$\tilde{l}(g) = \frac{1}{4}\gamma(l(g)).$$

When $g \notin H$ let

$$\tilde{l}(g) = \frac{1}{4}\gamma(l(g)) + \frac{1}{3}\gamma(\inf(l(gh) : h \in H)).$$

By the choice of l , H and γ the function \tilde{l} is an invariant length function.

It is now clear that for any $w \in D \cap N_1$ the value $\tilde{l}(\phi(w))$ is less than $\frac{1}{2}$. Moreover when $w \in D^3 \cap N$ then $\tilde{l}(\phi(w)) < \frac{10}{9}\varepsilon$, and when $w \in D \setminus N$ then $\tilde{l}(\phi(w)) \geq \frac{9}{40}$.

Note that when $g \in \phi(D) \setminus (H \cup N)$, the value $\tilde{l}(g)$ is greater than $\frac{9}{40} + \frac{3}{10} = \frac{21}{40}$, i.e. $> \frac{1}{2}$.

On the other hand if $F/N \models (wN = vN \cdot uN)$, where $w, v, u \in D$, then $w^{-1}vu \in N \cap D^3$, i.e. the corresponding distance between $\phi(w)$ and $\phi(v)\phi(u)$ is not greater than $\frac{10}{9}\varepsilon$. \square

We now describe some natural situations where functions $(l_c)_H^{*st}$ and $(l_c)_H^{ss't}$ appear. By Theorem 2.16 of [11] the conjugacy length l_c and the Hamming length are asymptotically equivalent in the class \mathcal{S} of all symmetric groups S_n . This in particular implies that any sofic group embeds into a metric ultraproduct of normed groups (S_n, l_c) . By Lemma 2.4 l_c can be replaced by any $(l_c)^{*st}$ or $(l_c)_H^{ss't}$ for fixed s, s', t and $H \in \{A_n, S_n\}$.

On the other hand functions of this kind can be applied for embeddings of sofic groups into some special metric ultraproducts. We now demonstrate it in the case of *LEF groups*.

Definition 4.3 ([5]) *A group G is called LEF if for every finite subset $D \subseteq G$ there is a finite group C and an injective map $\phi : D \rightarrow C$ such that any triple $h, g, hg \in D$ satisfies $\phi(hg) = \phi(h)\phi(g)$.*

We think that it is folklore that LEF is equivalent to statement (i) of the following proposition. We give the proof below just for a curious observation that LEF is equivalent to (ii) too. This is based on Theorem 4.3 of [7].

Proposition 4.4 *Let P be a finitely generated group. The group P is LEF if and only if one of the following equivalent conditions holds:*

(i) *any presentation $P = F/N$ so that N is a normal subgroup of a finitely generated free group F , satisfies the following property:*

for any two finite subsets $D_1 \subseteq N$ and $D_2 \subseteq F \setminus N$ there exists a normal subgroup $H < F$ of finite index so that $D_1 \subseteq H$ and $D_2 \cap H = \emptyset$;

(ii) *the (coloured) Cayley graph of P has the finite model property: any sentence of the first order theory of this graph holds in a finite graph.*

Proof. By Theorem 4.3 of [7] the conditions (i) and (ii) are equivalent. The condition (i) obviously implies LEF. To see that LEF implies (ii) we use the following consequence of Proposition 1.1 of [7]:

the Cayley graph Γ_P of a finitely generated group P has the finite model property if and only if for every natural n there is a finite graph Γ_n so that any point $v \in \Gamma_P$ and any $v' \in \Gamma_n$ have isomorphic n -balls.

Now note that when P is LEF then choosing the n -ball $B_n(1)$ of the neutral element 1 as a finite subset $D \subseteq P$ from the definition of LEF we find a finite group containing the partial group $B_n(1)$. Taking the subgroup generated by $B_n(1)$ if necessary, we obtain a finite group such that its Cayley graph has n -balls isomorphic to $B_n(1)$. We see that Γ_P has the finite model property. \square

Let P be a finitely generated group which is LEF. Let P_1 be a distinguished normal subgroup of P , say $P = F/N$ and $P_1 = N_1/N$, where F is a finitely generated free group and $N \triangleleft N_1 \triangleleft F$.

Definition 4.5 *A subgroup $P_1 < P$ is called LEF-separated if for any finite subsets $D_1 \subseteq N$ and $D_2 \subset F \setminus N$ there is a homomorphism ϕ from F to a finite group C so that $\phi(D_1) = \{e\}$, ϕ is injective on D_2 and*

$$\phi(D_2 \setminus N_1) \cap \phi(N_1) = \emptyset.$$

It is clear that if P_1 is of finite index, then P_1 is LEF-separated.

The following observation shows that when P is centreless and P_1 is LEF-separated we can embed P into a metric ultraproduct \hat{P} of normed groups introduced in Section 2 so that P_1 consists of all elements of P which have norms $\leq \frac{1}{2}$.

Proposition 4.6 *Let $N \triangleleft N_1 \triangleleft F$, where F is a finitely generated free group. Let $P = F/N$ be LEF, centerless and $P_1 = N_1/N$ be LEF-separated in P .*

*Then P embeds into a metric ultraproduct of finite groups with trivial centre and with metrics defined by pseudo length functions of the form $(l_c)_{\hat{H}}^{*st}$ so that P_1 consists of all elements of P which have norms $\leq \frac{1}{2}$.*

Proof. We want to verify the appropriate version of the approximation property for P where $\varepsilon = 0$, $\delta_g = \frac{1}{4}$ and the length of elements from P_1 is $\leq \frac{1}{2}$. Let D be a finite subset of F/N presented by words $\{w_1, \dots, w_k\} \subset F$. We may assume that the empty word belongs to D . For any $w \in D^3$ which does not present the neutral element find $w' \in F$ with non-trivial $[w, w']$ in F/N . Let D' consist of all words $w \in D^3$ and the corresponding $[w, w']$ of that form.

By LEF-separation there is a homomorphism ϕ from F to a finite group G so that $D' \cap N \subset \text{Ker}\phi$, $(D' \setminus N) \cap \text{Ker}\phi = \emptyset$ and $\phi(D' \setminus N_1) \cap \phi(N_1) = \emptyset$. Let $H = \phi(N_1)$.

Note that for any $h \in H$ the value $(l_c)_H^{*s1}(h)$ is less than $\frac{1}{2}$. Since for $w \in D^3 \setminus N$, the element $\phi(w)$ does not belong to the center of G , for any non-trivial $g \in \phi(D^3)$ the value $|g^G|$ is greater than 1. Thus we may choose s so that $(l_c)_H^{*s1}(g) > \frac{1}{4}$ for any non-trivial $g \in \phi(D^3)$. Thus $(l_c)_H^{*s1}(g) > \frac{1}{4}$ for all $g \in G$ and $(l_c)_H^{*s1}(g) > \frac{1}{2}$ for $g \notin H$.

On the other hand if $F/N \models (wN = vN \cdot uN)$, where $w, v, u \in D$, then $w^{-1}vu \in N \cap D'$, i.e. $\phi(w) = \phi(v)\phi(u)$.

□

Remark 4.7 It is worth noting that if in the proof above when we apply LEF-separation we can additionally obtain that for any $g \in \phi(D)$ the coset gH does not intersect $Z(G)$, we can replace functions $(l_c)^{*s1}$ by $(l_c)^{ss't}$ for appropriate s and s' . Indeed, as above we may choose s' so that $(l_c)_H^{ss'1}(g) > \frac{1}{4}$ for any non-trivial $g \in \phi(D^3)$. Moreover we can now choose s so that $(l_c)_H^{ss'1}(g) > \frac{1}{2}$ for $g \in \phi(D^3) \setminus H$.

References

- [1] G.Arzhantseva and L. Paunescu, Linear sofic groups and algebras, arXiv: 1212.6780.
- [2] M.Brandenbursky, S.R.Gal, J.Kendra, M.Marcinkowski, Cancellation norm and the geometry of biinvariant word metrics. ArXiv: 1310.2921.
- [3] J.Dieudonné, La géométrie des groupes classiques, (Springer, Berlin-Göttingen-Heidelberg, 1955)
- [4] G.Elek and E.Szabo, Hyperlinearity, eventually free actions and L^2 -invariants, Math. Ann. 332(2005), 421 - 441.
- [5] E.Gordon and A.Vershik, Groups that are locally embeddable in the class of finite groups, Algebra i Analiz 9(1997), no. 1 71 - 97. (Russian)
- [6] L.Glebsky and L.M.Rivera, Sofic groups and profinite topology on free groups, J.Algebra 320(2008), 3512 - 3518.
- [7] A.Ivanov, Cayley graphs having nice enumerations, Israel J. Math., 137(2003), 61 - 108.
- [8] M.Liebeck and A.Shalev, Diameters of finite simple groups: sharp bounds and applications, Ann. Math. 154(2) (2001), 384 - 406.

- [9] V.Pestov, Hyperlinear and sofic groups: a brief guide, *Bull. Symb. Logic*, 14(2008), 449 - 480.
- [10] F.Point, Ultraproducts and Chevalley groups, *Arch. Math. Log.*, 38(1999), 335 - 372.
- [11] A.Stolz and A.Thom, On the lattice of normal subgroups in ultraproducts of compact simple groups, *arXiv: 1207.0977*.
- [12] A.Thom, About the metric approximation of Higman's group, *J.Group Theory*, 15(2012), 301 - 310.
- [13] J.Wilson, On simple pseudofinite groups, *J. London Math. Soc. (2)*, 51 (1995), 471 - 490.

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